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UNDERGRADUATE MATHEMATICS CLUBS.

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CLUB ACTIVITIES.

THE MATHEMATICS AND PHYSICS CLUB OF THE UNIVERSITY OF ALABAMA, University, Ala.

This club was founded in November, 1916, under the name Mathematics Club, but at the second meeting this year the name was changed to that indicated above. The object of the club as stated in the constitution is "to encourage interest in mathematical and physical science at the University of Alabama and to facilitate the acquaintance of persons at the University who are interested in mathematics and physics." The average attendance at meetings now is about 12 although as many as 50 have been present at special meetings.

The officers for 1917-18 are: President, Robert F. Leftwich '18; secretary-treasurer, Donald H. Thornbury '18. The program committee consists of these officers and of Professor Tomlinson Fort, the head of the department of mathematics. The following programs were given during the autumn of 1917:

October 15: "Fourth Dimension" by David Skolnick '19; "Duplicating the Cube" by Robert T. Milne, instructor in mathematics.

November 7: "Curves in the Physical and Chemical Laboratories" by Maurice B. Amise '18; "John Napier" by Reginald C. Smith '19.

December 1: "Mathematics in Artillery Practice" by Robert F. Leftwich '18.

December 7: "Curved Flight of a Base-ball and similar Problems" by Donald H. Thornbury '18.

BARNARD COLLEGE MATHEMATICS CLUB, Columbia University, New York, N. Y.

This club was founded in 1909, in order "to foster an interest in mathematics." All students who are taking or who have had the equivalent of a course in analytics and elementary calculus are eligible for membership. There are now about 40 members and the average attendance is 25. The club is entirely in the hands of students who elect each year an honorary president from among the members of the mathematics faculty. This officer takes no part in the affairs of the club except in an advisory capacity.

The officers of the club for 1917-18 are: Honorary president, Dr. George W. Mullins; president, Ellen Leut '18; vice-president, Viola Williams '18; treasurer, Sophie Schulman '18; secretary, Mary Wellick '18; program committee: Fannie Rubenstein '18, Sophia Koerner '19 and Veronica Jentz '20.

Each November the club has its annual dance and once a year the alumnæ of the club are called together and the program is in the nature of an alumnæ report. The following programs were given in 1916-18:

October, 1916: "Dynamic Trajectories" by Professor Edward Kasner.

November: "Non-homogeneous Coördinates" by Eugenia Hausle '17.

- December: "Fourth-Dimensional Vistas"¹ by Beatrice Walker '17.
- February, 1917: "Non-Euclidean Geometry" by Helena Bausch '17; "Measurements of Time" by Sophie Schulman '18.
- March: "Mathematics of Navigation" by Harold Jacoby, professor of astronomy in Columbia University.
- April: "Paper Folding" by Mary Welleck '18; "Geometrical Puzzles" by Fanny Rubenstein '18.
- May: "Women in Science" by Professor Kasner; "Mathematical Fallacies" by Dr. Charles A. Fisher, instructor in mathematics; "Mathematics of Business" by Dr. George W. Mullins.
- October 15, 1917: Opening meeting. Informal addresses by Professor Frank N. Cole, and Dr. G. W. Mullins and Dr. Kenneth W. Lamson, instructors in mathematics.
- December 7: "Generalizations of Elementary Mathematical Concepts" by Professor Thomas S. Fiske.
- January 15, 1918: "An Introduction to Projective Geometry" by Gretchen Torek '19; "Mathematical Card Tricks" by Fanny Rubenstein '18; "Magic Squares" by Dorothy Jacobs '18.
- February 19: "Curve Families" by Dr. G. W. Mullins.
- March 15: "Correlation and Graphical Methods" by Helena Bausch '17 and Evelyn Davis '17, now in the research department of the American Telephone and Telegraph Company.
- April 30: "Infinite Series" by Joyce Buckbee, '18.

MATHEMATICS CLUB OF COLUMBIA UNIVERSITY, New York, N. Y.

In order "to stimulate and further the interests of mathematical scholarship amongst undergraduate students" and first-year graduate students at Columbia University the Junior Mathematical Colloquium was organized in November or December, 1910. Three years later this developed, under the supervision of Professor Herbert E. Hawkes, into the present more formal club with a constitution. The moving spirits in the founding of the colloquium were Professor Charles C. Grove and Dr. N. J. Lennes whose activity at the University of Chicago along similar lines has been already referred to in these columns.

The officers for 1917-18 are: President, William L. Schaaf '19; vice-president, Russell D. Burdick '19; secretary-treasurer, Francis W. Rogers '19. The executive committee (which acts as program committee) is composed of "officers of the club and a member of the Faculty of Columbia University nominated by the Mathematics Department of Columbia University." This faculty adviser is now Professor C. C. Grove. The attendance at the meetings last year varied from 9 to 32 with an average of about 19; the average attendance this year is about 14.

The following are programs of meetings 1916-18:

¹ *Four-Dimensional Vistas* is the title of a book by C. Bradgon (New York, A. A. Knopf, 1916).

- October 16, 1916: "Algebraic Methods of Solving Quadratic Equations" by Franklin Hollander '19.
- October 30: "Graphical Solution of Quadratic Equations" by Emil A. Goerlich '19.
- November 13: "Life of Emory McClintock, Columbia's Greatest Graduate in the Field of Mathematics" by Professor Thomas S. Fiske.
- November 27: "Geometric Constructions of the Cubic" by Professor Henry B. Mitchell.
- December 18: "Some Theorems on the Cubic" by Charles J. Hyman '17.
- January 8, 1917: "Discussion of the Kind of Points in a Plane necessary to carry out the Geometry of Euclid" by Lewi Tonks '17.
- February 19: "Why is a Conic?" by Richard Wagner, Jr. '18.
- March 5: "Logarithms in General of Complex Numbers" by William L. Schaaf '19.
- March 19: "Equal Area Maps" by Israel Koral '20.
- April 2: "Rotating Figures" by William Malisoff Gr.
- April 23: "The Path of a Projectile" by Dr. Charles A. Fischer, instructor in mathematics.
- October 29, 1917: "Descartes and his Theory of Equations" by Professor William B. Fite.
- November 12: "Sturm's Function" by George Dean '20; "Perspective Triangles" by Charles P. Davis '20.
- November 26: Problems proposed for solution.
- December 4: Informal discussion meeting. Solutions of problems set at the last meeting.
- December 17: "Quaternions" by Victor Schachtel '19.
- February 18, 1918: "Two Proofs of the Fundamental Theorem of Algebra" by Alfred M. Michaelis '21 and William L. Schaaf '19.
- March 4: "Theory of Transfinite Numbers" (Cantor) by Franklin Hollander '19.
- March 18: "Mathematics in the Science of Physics" by Leon Morris '19.
- March 25: "Dimensionality" by Israel Koral '20.
- April 8: "General Theory of Relations" by Professor Cassius J. Keyser.
- April 22: "Paradoxes of the Infinite"¹ by Moses Davis '20.

THE EUCLIDEAN CIRCLE OF INDIANA UNIVERSITY, Bloomington, Ind.

This club was organized in September, 1907. In 1916-17 there were about 25 members and the attendance at each meeting was 10-12. During the current year it was not found feasible to reorganize the Circle.

Those qualified for membership have been students majoring in mathematics, juniors, seniors, graduates, and members of the mathematical faculty. The object of the organization was to make its members "better acquainted with the history of mathematics." "A club picture has been taken each year."

¹ B. Bolzano's *Paradoxien des Unendlichen* (first published in 1850, after his death; facsimile reprint: Berlin, Mayer und Müller, 1889) has been more than once the basis of lectures by Schwarz at the University of Berlin.

THE MATHEMATICAL AND PHYSICAL SOCIETY OF THE UNIVERSITY OF TORONTO,
Toronto, Ontario.

Among the existing undergraduate clubs of America this one is notable in more ways than one. In the first place it is the most venerable, since its thirty-sixth birthday is already a thing of the past. In the second place it has published three slight volumes of its proceedings.¹ The society was founded on January 27, 1882, when the report of a committee appointed to draw up a constitution was adopted and the following officers were elected: President, W. S. Loudon Gr.; vice-president, J. M. Clark; secretary-treasurer, D. Burns; corresponding secretary, T. G. Campbell; councillors: Faculty of Arts, A. H. McDougall (4th year), G. I. Riddell (3d year), T. Mulvey (2d year), G. H. Hogarth (1st year); School of Practical Science, D. Jeffrey (3d year), — Fotheringham (2d year). Of these officers Mr. Loudon is now professor of mechanics in the University of Toronto; Mr. Clark is a well-known lawyer of Toronto; Mr. McDougall is principal of the Collegiate Institute at Ottawa and Mr. Mulvey is Under Secretary of State at Ottawa.

A few years later when the students in the School of Practical Science became sufficiently numerous they seceded and formed the Engineering Society.

At first a graduate was chosen president, the other officers being undergraduates. Some years ago the constitution was changed, so that a graduate was to be honorary president and undergraduates were to hold the active offices. The officers for 1917-18 are as follows: Honorary president, Dr. John Satterly, assistant professor of physics; president, Norris E. Sheppard '18; vice-president, William W. Shaver '19; secretary-treasurer, Everett O. Hall '19; corresponding secretary, Hazel C. Miller '18; representatives of classes: Harry E. Foreman '18, Mary M. Stephens '19, William S. Vaughn '20, Donald F. Shugart '21.

The object of the society is "the encouragement of study and original research

¹ Since the editions of these publications were very small, and long since exhausted, and since particulars concerning them are lacking in all mathematical bibliographies, it may be well to put some details on record here. The title page of each part: *Papers read before the Mathematical and Physical Society of Toronto University during the year(s) 1890-91 [1891-92] (1892-93, 1893-94)*, Toronto: Rowsell and Hutchison, Printers, King Street East, 1891 [1892] (1895). Pages 60 [60] (54).

Contents, *Part 1*: A. Baker, "Poetic Interpretation in Mathematics," pp. 7-21; A. T. De Lury, "On Certain Deductions from the Theorem of Dr. Graves," pp. 22-30; A. C. Chant, "The Structure of Matter," pp. 31-42; F. Sanderson, "The Law of Human Mortality and its Place in Science," pp. 43-54; R. Henderson, "Newton's Laws of Motion," pp. 55-59. *Part 2*: W. J. Loudon, "Musical Scales, Their Origin, Formation, and the Physical Relation Which They Bear to Music," pp. 5-18; I. E. Martin, "The Religion of Algebraic Curves," pp. 19-32; C. A. Chant, "The Wave Theory of Sound," pp. 33-47; W. Gillespie, "Some Trigonometrical Expansions," pp. 48-53; G. R. Anderson, "Measurement of Time," pp. 54-60. *Part 3*: "On the Non-Euclidean Geometry," pp. 6-28 [a translation by A. T. DeLury of the "Note sur la géométrie non-euclidienne" in the *Traité de Géométrie* of Rouché et de Comberousse]; J. C. Glashan, "Elementary Proof of a Theorem in Trigonometry," pp. 29-32; C. A. Chant, "The Development of Electricity," pp. 33-46; A. M. Scott '96, "Geometrical Representation of the Special Roots of Unity" (abstract), pp. 47-49; Miss J. S. Hillock '95, "Laplace" (abstract), p. 50; G. W. Rudlen '94, "The Quadrature of the Circle" (abstract), pp. 51-52; Miss A. Lindsay '93, "Maria Gaetana Agnesi" (abstract), pp. 53-54.

in mathematics and physics, and the preservation of the results of such work." All the special students in mathematics and physics are considered members—about 75 now—and the average attendance at meetings is about 50.

"In earlier years the solution of problems was a feature of each program but that has been discarded and light refreshments have been introduced." The programs¹ for 1916-18 are as follows:

November 2, 1916: Opening meeting, social; "Aircraft," an address by John C. McLennan, professor of physics.

November 16: "Cultural Advantages of the Mathematics and Physics Course" by Herbert R. Rowan '17 ("To strive, to seek, to find, and not to yield");² "Science versus Art," Discussion led by Mabel Campbell '19 and James B. Russell '19 ("For, even though vanquished, he could argue still").

November 30: "Our Society's Past" by Clarence A. Chant, associate professor of astrophysics ("Lives of great men all remind us"); "Experiments on Probability" by Norris E. Sheppard '18.

December 14: "Newton" by Aylmer B. Paisley '20 ("And still the wonder grew"); "The Moon: a look at our next neighbor" by Janet M. Halliday '18.

January 11, 1917: "Societies to which mathematics and physics students may aspire" by Professor John C. Fields ("Held from afar, aloft, th' immortal prize And urged the rest, by equal steps to rise"); "Experiments on Light" by Albert R. Self '17.

January 25: "Euclid" by William W. Shaver '19 ("Let us begin and carry up this corpse, Singing together"); "Mathematical Symbolism" by James C. Thompson '18 ("Earth hath not anything to show more fair").

February 8: Open Meeting ("Up, up my friend and quit your books").

February 22: "Life Assurance" by William A. Jackson '17 ("Yet the strong man must go"); "Experiments in Heat and Mechanics" by Professor John Satterly ("While we sat wrapt in wonder").

March 8: Annual Business Meeting ("The old order changeth, Yielding place to new").

November 8, 1917: Opening meeting. "George Russell" by Professor Alfred T. DeLury.

November 22: "The ancient Mathematicians" by Douglas H. Blatchford '18; "Germany and the Germans" by Professor John C. Fields.

December 6: Graduates' meeting. "Life in an Actuary's Office" by Janet Holmes '17; "Mathematicians of the Middle Ages" by William A. Jackson '17; "Spectra" by Florence M. Quinlan '17.

December 13: Social meeting.

January 10, 1918: "Mathematics of Modern Times" by Mabel C. Childs '18; "J. J. Thompson" by Ernest R. I. Pratt '18.

January 27: Debate between First and Second Year Students.

¹ For many years programs have been issued annually in printed form.

² After most of the titles on the printed program for 1916-17 were quotations, such as this, in small type.

February 9: Open meeting. "Eclipses" by Professor Clarence A. Chant.
 February 21: "Faraday" by Ila B. Giles '19; "The Mathematics and Physics Courses at the Royal College of Science, London" by Professor John Satterly.
 March 7: Annual Business Meeting.

MATHEMATICS CLUB, The Western College for Women, Oxford, Ohio.

This club was organized in 1905 "to stimulate interest in certain phases of mathematics which, while closely related to class work, do not fall directly under it. The members include those in the elective mathematics classes and the teaching staff of the department. For the current year there are 25 student and 3 faculty members. There are no officers; the seniors take turns in looking after the meetings. The programs for the past three years are as follows:

December 6, 1915: "Some Constructions leading to Conic Sections" by Frances Orr, instructor in mathematics; "A Problem in Factoring" by Kathleen Banker '16; "Parallelograms inscribed in a Rectangle" by Mary C. Little '16; "A few Classic Unknowns in Mathematics," report of G. A. Miller's article in *Scientific Monthly*, October, 1915, by Helen McBride '16.

February 14, 1916: "Higher Plane Curves, their use in Mechanics" by Harriet Rice '16; "Higher Plane Curves, their Application to Geometrical Constructions" by Ginevra E. McCoy '17; "Mathematics in Dr. Eliot's Five-Foot Shelf of Books," report of W. H. Bussey's article in this MONTHLY, June, 1915, by Norrine DeLaney '17; Current Topics.

March 13: "Number Systems of the North American Indians"¹ by Ethel S. Sebald '17; "Early History of Mathematics in the United States" by Marie Pearson '17.

October 23, 1916: Picnic.

November 20: "The Triangle and its Circles"² by Edna P. Pepper '18; "A Circle Theorem"³ by Anna S. Armstrong '17; "Elementary Proof of a Theorem on the Circle due to F. Morley," report of T. Dantzig's article in this MONTHLY, September, 1916, by Mary C. Little, assistant in mathematics; "Recent additions to the mathematics library" by Annie C. Crane '19 and Edna Berkele '19.

January 22, 1917: "Greek Methods of Solving Quadratic Equations"⁴ by Ethel S. Sebald '17; "Some modern Solutions of the Quadratic" by Anna S. Armstrong '17; "Graphic Solution of $y = x^n$ " by Mary R. Shipp '18; Report on

¹ W. C. Eells's paper with this title appeared in this MONTHLY, November and December, 1913.

² The following pamphlet on this subject has been published: W. H. Bruce, *Some Noteworthy Properties of the Triangle and its Circles* (Heath's Mathematical Monographs, No. 8). Boston, Heath, 1903, pp. 28.

³ R. A. Johnson's article with this title appeared in this MONTHLY for May, 1916. It was followed by A. Emch's "Remarks on the Foregoing Circle Theorem." The theorem in question ("If three equal circles are drawn through a point, the circle through their intersections is equal to each of them") was announced by Michail Manoilescu in *L'Education Mathématique*, 1 Mai, 1910, 12 année, p. 128.

⁴ W. C. Eells's paper with this title was published in this MONTHLY, January, 1911.

"Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi"¹ by Ginevra E. McCoy '17; Current Topics by Mary E. Spencer '19, Rhoda E. Trook and Dorothy M. Wilkinson '19.

March 5: "On the Representation of Large Numbers and Infinite Processes," report of A. Emch's article in *Scientific Monthly*, March, 1916, by Jessie P. Wise '18; "Rithmomachia and other number games"² by Sarah M. Sloan '18; "Perfect Numbers" by Ginevra E. McCoy '17; "A Curious Convergent Series"³ by Mary C. Little, assistant in mathematics.

October 22, 1917: Reception for Dr. Ettalene Grice '08, who is assisting in reading some of the Babylonian tablets in the Yale Museum; she gave a talk on the "Mathematics of the Babylonians."

October 29: "Magic Squares" by Helen T. Anger '19 and Sena M. Sutherland '18; "Some Peculiar Properties of Numbers" by Edna M. Sebald '18.

November 26: "Common Geometric Forms in Art" by Annie C. Crane '19 and Jessie P. Wise '18; Current Topics by Helen B. Griesmer '20.

February 11, 1918: "The Three Famous Problems of Antiquity": (a) "The Trisection of an Angle" by Mary E. Thomas '18; (b) "The Duplication of the Cube" by Mary R. Shipp '18; (c) "The Quadrature of the Circle" by Sarah M. Sloan '18.

March 4: "Certain Typical Problems, their Origin and History" by Edna P. Pepper '18 and Mary E. Spencer '19; Current Topics by Edith M. Sawin '19.

April 15: "The Fourth Dimension" by M. Lucile Brown, instructor in mathematics.

"At the close of each program a little time is devoted to informal discussion and a cup of tea." At Western College it is felt that even the few meetings of the club held each year "give a tremendous impetus to the genuine interest in the work of the department."

TOPICS FOR CLUB PROGRAMS.

Addendum: In the last line of Topic 5 add, after *L'Education Mathématique*, 1913, Vol. 15, pp. 161-163; Vol. 16, pp. 1-3, 13-15.

9.4 GOLDEN SECTION.

In the *Elements* of Euclid (who flourished about 300 B. C.), the following propositions occur: (1) "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment" (Book II, proposition 11); (2) "To cut a given finite line in extreme

¹ With an Introduction, Critical Notes and an English Version by L. C. Karpinski, New York, Macmillan, 1915.

² The *Teachers College Record* for November, 1912 (Vol. 13, pp. 385-495), is devoted to articles on "Number Games and Number Rhymes" by D. E. Smith, C. W. Hunt, F. J. Flynn, C. C. Eaton, R. K. Atwell and F. B. Selkin. Chapter 3 (pp. 413-422) on "Rithmomachia, the great medieval number game" by D. E. Smith and C. C. Eaton is reprinted, with a few modifications, from the *AMERICAN MATHEMATICAL MONTHLY*, April, 1911. A bibliography is given on page 495.

³ F. Irwin's paper with this title appeared in this *MONTHLY*, May, 1916.

⁴ The reader is reminded of the statement introducing Topic 8 in the last issue of the *MONTHLY*; topics 8, 9 and 10 were selected from those which arise in the discussion of various forms of growth.

and mean ratio" (Book VI, proposition 30).¹ While these propositions are equivalent in statement the methods of construction given by Euclid are quite different. There can be little doubt that the construction in the second is due to Euclid and in the first to the Pythagoreans (fifth century B. C.). The result is used "To construct an isosceles triangle having each of the angles at the base double of the remaining one" (*Elements*, Book IV, 10) and this leads to the construction of a regular pentagon (Book IV, 11).

In the *Elements*, book XIII, the first five propositions, which are preliminary to the construction and comparison of the five regular solids, and deal with properties of a line segment divided in extreme and mean ratio, are usually attributed to Eudoxus who flourished about 365 B. C. Proclus tells us that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section"; scholars agree that "the section" refers to the division in extreme and mean ratio.

The so-called book XIV of Euclid's *Elements*, written by Hypsicles of Alexandria between 200 and 100 B. C., contains some results concerning "the section."

In recent times the name golden section has been applied to the division of a line segment as above² in the ratio $(\sqrt{5} - 1) : 2$. Terquem believed that the expression "extreme and mean ratio" (which is an exact translation of Euclid's Greek phrase) is "une réunion de mots ne présentant aucun sens,"³ and following J. F. Lorenz (1781) employed the term "continued section." Terquem has also suggested:⁴ "diviser une droite décagonalement." Leslie introduced the term "medial section."⁵ "Divine proportion" was used by Fra Luca Pacioli in 1509⁶ and possibly earlier by Pier della Francesca;⁷ "sectio divina" and "proportio divina" occur in the writings of Kepler.

¹ These enunciations are taken from *The Thirteen Books of Euclid's Elements* translated with introduction and commentary by T. L. Heath, 3 vols., Cambridge, at the University Press, 1908. For statements in connection with our discussion see particularly, Vol. 1, pp. 137, 403; Vol. 2, p. 99; Vol. 3, p. 441.

² The earliest instances which I find of the use of the term golden section are in J. Helmes, "Eine einfachere, auf einer neuen Analyse beruhende Auflösung der sectio aurea, nebst einer kritischen Beleuchtung der gewöhnlichen Auflösung und der Betrachtung ihres pädagogischen Werthes." *Archiv der Mathematik*, Grunert, Band 4, 1844, pp. 15-22; in A. Wiegand, *Geometrische Lehrsätze und Aufgaben*, Band 2, 1. Abtheilung, Halle, 1847, p. 142; and also in A. Wiegand, *Der allgemeine goldene Schnitt und sein Zusammenhang mit der harmonischen Theilung* . . . Halle, 1849.

³ *Nouvelles annales de mathématiques*, Paris, tome 12, 1853, p. 38.

⁴ *Journal de mathématiques pures et appliquées*, Paris, tome 3, 1838, p. 98.

⁵ J. Leslie, *Elements of geometry, geometrical analysis and plane trigonometry*, Edinburgh, 1809, p. 66.

⁶ *Divina proportione opera a tutti gli ingegni perspicaci e curiosi necessaria que ciaseum studioso di philosophia: prospettiva, pictura, sculptura, architectura: musica: e altre matematiche . . . Venetiis . . . 1509*. Although not printed till 1509 the manuscript of this work was completed in 1497. The geometrical drawings were made by Leonardo da Vinci. Another edition of the Latin text "herausgegeben, übersetzt und erläutert von C. Winterberg" appeared at Vienna (Gräser) 1889. Another edition 1896, 6 + 367 pp. A full analysis of Pacioli's work is to be found in A. G. Kästner, *Geschichte der Mathematik* . . . Band I, Göttingen, 1796, pp. 417-449. See also M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Band 2, 2. Auflage, Leipzig, 1900, pp. 341 ff., 347.

⁷ It has been shown by G. Mancini that parts of Pacioli's *Divina proportione* were taken from a Vatican manuscript by Pier della Francesca. See (1) G. Pittarelli, *Atti del IV. Congresso*

Pacioli's work was doubtless influential in inspiring a certain amount of mysticism in the consideration of golden section by later writers. In a work published in 1569, P. Ramus associates the Trinity with the three parts of golden section. A little later Clavius wrote of its "god-like proportions." As noted above Kepler declared himself similarly. He said also: "Geometry has two great treasures, one is the Theorem of Pythagoras, the other the division of a line into extreme and mean ratio; the first we may compare to a measure of gold, the second we may name a precious jewel."¹

There is an interesting passage on golden section by Albert Girard in his edition of Stevin's works.² Girard gives a method of expressing the ratio of the segments of a line (cut in golden section) in rational numbers that converge to the true ratio. For this purpose he takes the sequence

$$(1) \quad 0, 1, 1, 2, 3, 5, 8, 13, 21, \dots,$$

every term of which (after the second) is equal to the sum of the two terms that precede it, and says, after Kepler, any number in this progression has to the following the same ratios (nearly) that any other has to that which follows it. Thus 5 has to 8 nearly the same ratio that 8 has to 13; consecutive numbers such as 8, 13, 21 nearly express the in golden section. Since the fractions

$$(2) \quad \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \dots$$

are the various convergents of the continued fraction

$$\frac{\sqrt{5}-1}{2} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 \dots}}},$$

Maupin reasons with force (after taking into account all which follows in the note) that Girard was probably familiar with the elements of continued fractions. Simson interprets Girard's reasoning differently. In notes on the next topic I shall have occasion to return to the remarkable series (1) and (2).

dei matematici, tomo 3, Roma, 1909; (2) G. Mancini, "L'opera 'De Corporibus Regularibus' di Pietro Franceschi detto Francesca usurpata da Fra Luca Pacioli" (con dodici tavole) *Reale accademia dei Lincei*, 1915. See review by F. Cajori in this MONTHLY, Vol. 23, 1916, p. 384. (3) G. B. de Toni, "Intorno al codice sforzesco 'De divina proportionibus' di Luca Pacioli e i disegni geometrici di quest'opera attribuiti a Leonardo da Vinci," *Modena Soc. dei naturalisti e matematici*, atti, 13, 1911, pp. 52-79.

¹ Exact references to sources, and some quotations from originals, are given in (1) J. Tropicke, *Geschichte der Elementar-Mathematik*, Band 2, Leipzig, Veil, 1903; (2) F. Sonnenburg, *Der goldene Schnitt. Beitrag zur Geschichte der Mathematik und ihre Anwendung*. (Progr.). Bonn, 1881. (Not always reliable).

² *Les œuvres mathématique de Simon Stevin* revues, corrigées et augmentées par A. Girard. Leyde, 1634, pp. 169-170. The passage in question is reprinted with commentary in G. Maupin, *Opinions et curiosités touchant la mathématique* (deuxième série), Paris, 1902, pp. 203-209. It has been discussed also by R. Simson, *Philosophical Transactions*, 1753, Vol. 48, pp. 368-377.

In the nineteenth century the literature of golden section is by no means inconsiderable. It includes at least a score of separate pamphlets and books and many times that number of papers. In numerous, voluminous and rather unscientific writings A. Zeising¹ finds golden section the key to all morphology and contends, among other things, that it dominates both architecture and music. A distinctly new line was set under way by Fechner² who applied scientific experimental method to the study of æsthetic objects.³ He was led to the conclusion that the rectangle of most pleasing proportions was one in which the adjacent sides are in the ratio of parts of a line segment divided in golden section.³

Sir Theodore Cook discusses⁴ golden section from some new points of view in connection with art and anatomy, and the writings of F. X. Pfeifer⁵ remind one both in subject matter and style of treatment of Zeising's publications.

For mathematical treatment of problems in golden section, in ordinary or generalized form, see also the papers by C. Thiry⁶ and R. E. Anderson,⁷ E. Catalan's *Théorèmes et Problèmes de géométrie élémentaire*⁸ and Emsman's program⁹ containing more than 350 relations and problems.

10. A FIBONACCI SERIES.

Foremost among mathematicians of his time was Leonardo Pisano (also known as Fibonacci) who flourished in the early part of the thirteenth century. His greatest work is *Liber abbaci* "a Leonardo filio Bonacci compositus, anno 1202 et correctus ab eodem anno 1228." It was first printed in 1857.¹⁰

¹ For example (1) *Neue Lehre von den Proportionen des menschlichen Körpers aus einem bisher unerkant gebliebenen, die ganze Natur und Kunst durchdringenden morphologischen Grundgesetze entwickelt*, Leipzig, 1854, 457 pp.; (2) *Das Normalverhältnis der chemischen und morphologischen Proportionen*, Leipzig, 1856, 114 pp. and the posthumous work: (3) *Der goldene Schnitt*, Leipzig, 1884, 28 pp. Cf. S. Günther, "Adolph Zeising als Mathematiker" *Zeitschrift für Mathematik und Physik*, Historisch-literarische Abtheilung, Band 21, 1876, pp. 157-165.

² G. T. Fechner, *Zur experimentalen Aesthetik*, Leipzig, 1871.

³ C. L. A. Kunze speaks of "Rechteck der schönsten Form" in his *Lehrbuch der Planimetrie*, Weimar, 1839, p. 124. A reference may be given to a recent discussion of "printer's oblong" and "golden oblong" in H. L. Koopman, "Printing Page Problems with Geometric Solutions," *The Printing Art*, Cambridge, Mass., 1911, Vol. 16, pp. 353-356.

⁴ T. A. Cook, *The Curves of Life*, London, Constable, 1914.

⁵ (a) "Die Proportion des goldenen Schnittes an den Blättern und Stengeln der Pflanzen," *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 1885, Vol. 15, pp. 325-338; (b) *Der goldene Schnitt und dessen Erscheinungsformen in Mathematik Natur und Kunst*, Augsburg, [1885], 230 pp. A resumé of this work given by O. Willman in *Lehrproben und Lehrgänge aus der Praxis der Gymnasien und Realschulen*, 1892 was the basis of E. C. Ackermann, "The Golden Section," *AMERICAN MATHEMATICAL MONTHLY*, 1895, Vol. 2, pp. 260-264. Cf. *Zeitschrift f. math. und naturwiss. Unterricht*, 1887, Vol. 18, pp. 44-47, 605-612.

⁶ C. Thiry, "Quelques propriétés d'une droite partagée en moyenne et extrême raison," *Mathesis*, 1894, Vol. 14, pp. 22-24.

⁷ "Extension of the Medial Section Problem and Derivation of a Hyperbolic Graph," *Proceedings of the Edinburgh Mathematical Society*, 1897, Vol. 15, pp. 65-69.

⁸ 6e éd., Paris, 1879, pp. 261-263. Some of these properties are given in the first edition of this work by H. C. de La Frémoire, Paris, 1844.

⁹ *Zur sectio aura*. Progr. Stettin, 1874 (Cf. *Zeitschrift f. math. und naturw. Unterricht*, Vol. 5, pp. 289-291).

¹⁰ *Il liber Abbaci di Leonardo Pisano* pubblicato da Baldassure Boncompagni, Roma, MDCCCLVII. For an analysis of this work see M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Band II, 3. Auflage, Leipzig, Teubner, 1900, pp. 5-35.

Among miscellaneous arithmetical problems of the twelfth edition is one entitled "How many pairs of rabbits can be produced from a single pair in a year."¹ It is supposed (1) that every month each pair begets a new pair which, from the second month on, becomes productive; and (2) that deaths do not occur. From these data it is found that the number of pairs in successive months would be as follows:

(3) 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377.

These numbers follow the law that every term after the second is equal to the sum of the two preceding and form, according to Cantor, the first known recurring series in a mathematical work. The doubtful accuracy of this latter statement has been pointed out by Günther.²

The series (3) was well known to Kepler who discusses and connects it with golden section and growth, in a passage of his *De nive sexangula*, 1611.³ Commentaries of Girard and Simson, and the relation of the series to a certain continued fraction, have been noted above in connection with Topic 9. But the literature of the subject is very extensive and reaches out in a number of directions. In what follows u_n will be regarded as the $(n+1)$ st term of what we shall call the Fibonacci Series (1); so that $u_0 = 0$ and $u_1 = 1$. For reasons which shall appear later the terms Lamé series, and Braun or Schimper-Braun series have been employed in this connection. Girard observed that the three numbers u_n, u_{n+1}, u_{n+1} ⁴ may be regarded as corresponding to lengths which form an isosceles triangle of which the angle at the vertex is very nearly equal to the angle of the regular pentagon.

The relation $u_{n-1}u_{n+1} - u_n^2 = (-1)^{n+1}$ while implied in the passage quoted from Kepler is more explicitly used by Simson (1753). It was to this relation, and hence to the Fibonacci series that Schlegel⁵ was led when he sought to generalize the well-known geometrical paradox of dividing a square 8×8 into four parts which fitted together form a rectangle 5×13 .⁶ Catalan found (1879) the more

¹ Pages 283-284.

² S. Günther, *Geschichte der Mathematik*, 1. Teil, Leipzig, Göschen, 1908, p. 137.

³ J. Kepler, *Opera*, ed. Frisch, tome 7, pp. 722-3. After discussions of the form of the bees' cells and of the rhombo-dodecahedral form of the seeds of the pomegranite (caused by equalizing pressure) he turns to the structure of flowers whose peculiarities, especially in connection with quincuncial arrangement he looks upon as an emanation of sense of form, and feeling for beauty, from the soul of the plant. He then "unfolds some other reflections" on two regular solids the dodecagon and icosahedron "the former of which is made up entirely of pentagons, the latter of triangles arranged in pentagonal form. The structure of these solids in a form so strikingly pentagonal could not come to pass apart from that proportion which geometers to-day pronounce divine." In discussing this divine proportion he arrives at the series of numbers 1, 1, 2, 3, 5, 8, 13, 21 and concludes: "For we will always have as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost. I think that the seminal faculty is developed in a way analogous to this proportion which perpetuates itself, and so in the flower is displayed a pentagonal standard, so to speak. I let pass all other considerations which might be adduced by the most delightful study to establish this truth."

⁴ There is a typographical error (13 for 21) in Girard's discussion in this connection.

⁵ V. Schlegel, "Verallgemeinerung eines geometrischen Paradoxons," *Zeitschrift für Mathematik und Physik*, 24. Jahrgang, 1879, pp. 123-128.

⁶ This paradox was given at least as early as 1868 in *Zeitschrift für Mathematik und Physik*, Vol. 13, p. 162. Cf. W. W. R. Ball, *Mathematical Recreations and Essays*, 5th edition, London, Macmillan, 1911, p. 53; and E. B. Escott, "Geometric Puzzles," *Open Court Magazine*, Vol. 21, 1907, pp. 502-5.

general relation⁴ $u_{n+1-p}u_{n+1+p} - u_{n+1}^2 = (-1)^{n+2-p}(u_p)$,¹ from which may be derived $u_{n+1}^2 + u_n^2 = u_{2n+1}$ first given, along with many other properties, by Lucas,² in a paper showing the relation between the Fibonacci series and Pascal's arithmetical triangle. It was Binet³ who showed that

$$2^n \sqrt{5} u_n = (1 + \sqrt{5})^n - (1 - \sqrt{5})^n,$$

and Catalan gave the following result a few years later:⁴

$$2^{n-1}u_n = \frac{n}{1} + 5 \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + 5^2 \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

Lucas showed the importance of the Fibonacci series in discussions of (a) the decomposition of large numbers into factors and (b) the law of distribution of prime numbers.⁵ Binet was led to the series in his memoir on linear difference equations (*l. c.*), and Lamé indicated its application⁶ in determining an upper limit to the number of operations made in seeking the greatest common divisor of two integers. Landau evaluated the series $\Sigma(1/u_{2h})$ and $\Sigma(1/u_{2h+1})$, and found that the first was related to Lambert's series and the second to the theta series.⁷

For further references and mathematical discussions one may consult (1) *L'Intermédiaire des mathématiciens*, 1900, pp. 172-7; 1902, p. 43; 1915, pp. 39-40; (2) "Sur une généralisation des progressions géométriques," *L'Education mathématique*, 1914, pp. 149-151, 157-158; and (3) V. Schlegel, "Séries de Lamé supérieures," *El progreso matematico*, 1894, año 4, pp. 171-174.

As to growths it is particularly in connection with older chapters on leaf

¹ E. Catalan, *Mélanges mathématiques*, tome 2, [Liège, 1887], p. 319.

² E. Lucas, "Note sur la triangle arithmétique de Pascal et sur la série de Lamé," *Nouvelle correspondance mathématique*, tome 2, 1876, p. 74.

³ J. P. M. Binet, "Mémoire sur l'intégration des equations linéaires aux différences finies d'un ordre quelconque, à coefficients variables," *Comptes rendus de l'académie des sciences de Paris*, tome 17, 1843, p. 563.

⁴ *Manual des candidats à l'École Polytechnique*, tome 1, Paris, 1857, p. 86.

⁵ E. Lucas, (a) "Recherches sur plusieurs ouvrages de Léonard de Pise et sur diverses questions d'arithmétique supérieure. Chapter 1. Sur les séries récurrentes," *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche*, tome 10, pp. 129-170, Marzo, 1877; (b) Théorie des fonctions numériques simplement périodiques," *American Journal of Mathematics*, Vol. 1, 1878, pp. 184-229, 289-321 [on p. 299 are given the first 61 terms of the Fibonacci series and the factors of every term]; (c) "Sur la théorie des nombres premiers" [dated mai 1876], *Atti della r. Accademia delle Scienze di Torino*, Vol. 11, 1875-76, pp. 928-937; (d) "Note sur l'application des séries récurrentes à la recherche de la loi de distribution des nombres premiers," *Comptes rendus de l'académie des sciences*, Vol. 82, 1876, pp. 165-167. See also A. Aubry, "Sur divers procédés de factorisation," *L'Enseignement Mathématique*, 1913, especially §§ 11, 16 and 17, pp. 219-223.

⁶ B. Lamé, "Note sur la limite du nombre des divisions dans la recherche du plus grand commun diviseur entre deux nombres entiers." *Comptes rendus de l'académie des sciences*, tome 19, 1844, pp. 867-870. See also J. P. M. Binet, *idem*, pp. 939-941.

Because of results obtained in the above-mentioned memoir the Fibonacci series is frequently called the Lamé series. I can find no verification of Thompson's statement (*On Growth and Form*, p. 643) that the series 2/3, 3/5, 5/8, 8/13, 13/21, . . . "is called Lami's series by some, after Father Bernard Lami, a contemporary of Newton's, and one of the co-discoverers of the parallelogram of forces." Indeed the statement is doubtless incorrect.

⁷ E. Landau, "Sur la série des inverses des nombres de Fibonacci," *Bulletin de la société mathématique de France*, tome 27, 1899, pp. 298-300.

arrangement or phyllotaxis that the Fibonacci Series comes up. Among the earliest and most important of these are the memoirs of Braun (based on researches of Schimper and himself),¹ and L. et A. Bravais.³ Of later papers there are those by Ellis,³ Dickson,⁴ Wright,⁵ Airy,⁶ Günther,⁷ and Ludwig.⁸ Much that was fanciful and mysterious was swept away by the publication of P. G. Tait's note "On Phyllotaxis."⁹ Of recent books on the subject the most notable are those by Church,¹⁰ Cook,¹¹ and Thompson.¹² The two former are beautifully illustrated. The latter reproduces Tait's discussion in an appreciative manner.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Mr. E. S. LANE, fellow at the University of Chicago, has been appointed instructor in mathematics at Rice Institute.

Mr. W. G. SIMON, fellow at the University of Chicago, has been appointed instructor in mathematics at Western Reserve University.

Dr. OLIVE C. HAZLETT, instructor at Bryn Mawr College, has been appointed to an assistant professorship of mathematics at Mount Holyoke College.

¹ A. Braun, "Vergleichende Untersuchung über die Ordnung der Schuppen an den Tannenzapfen als Einleitung zur Untersuchung der Blätterstellung überhaupt," *Nova Acta Acad. Caes. Leopoldina*, Vol. 15, 1830, pp. 199-401.

² L. et A. Bravais, (1) "Sur la disposition des feuilles curvisériées," *Ann. des sc. nat.*, 2e série Vol. 7, 1837, pp. 42-110; (2) *Mémoire sur la disposition géométrique des feuilles et des inflorescences*, Paris, 1838.

³ R. L. Ellis, *Mathematical and Other Writings*, Cambridge, 1863; "On the Theory of Vegetable Spirals," pp. 358-372.

⁴ A. Dickson, "On some abnormal cases of pinus pinaster," *Transactions of the Royal Society of Edinburgh*, Vol. 26, 1871, pp. 505-520.

⁵ C. Wright, "The uses and origin of the arrangements of leaves in plants" (read 1871), *Memoirs of the American Academy*, Vol. 9, part 2, Cambridge, Mass., p. 384f.

⁶ H. Airy, "On Leaf Arrangement," *Proceedings of the Royal Society of London*, Vol. 21, 1873, pp. 176-179.

⁷ S. Günther, "Das mathematische Grundgesetz im Bau des Pflanzenkörpers," *Kosmos*, II. Jahrgang, Band 4, 1879, pp. 270-284.

⁸ F. Ludwig, "Einige wichtige Abschnitte aus der mathematischen Botanik," *Zeitschrift für mathematischen und naturwiss. Unterricht*, Band 14, 1883, p. 161f.

⁹ P. G. Tait, *Proc. Royal Society Edinburgh*, Vol. 7, 1872, pp. 391-4.

¹⁰ A. H. Church, *On the Relation of Phyllotaxis to Mechanical Laws*, London, Williams and Norgate, 1904. On page 5 Church writes: "The properties of the Schimper-Braun series 1, 2, 3, 5, 8, 13, . . . , had long been recognized by mathematicians (Gerhardt, Lamé). . . ." In *Botanisches Centralblatt*, Band 68, 1896, F. Ludwig writes (on p. 7) that the numbers of this series "werden vielfach von Botanikern als Braun'sche, von Mathematikern als Gerhardt'sche oder Lamé'sche Reihe bezeichnet." I have not been able to verify that any mathematician used the term Gerhardt series in this connection, or that anyone by the name of Gerhardt wrote about the Fibonacci series. From what has been indicated above it seems certain that "Gerhardt'sche" should be "Girard'sche."

¹¹ T. A. Cook, *The Curves of Life*, London, Constable, 1914.

¹² D'A. W. Thompson, *On Growth and Form*, Cambridge: at the University Press, 1917.